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# Time dependent black holes and thermal equilibration

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Abstract: We study aspects of a recently proposed exact time dependent black hole solution of IIB string theory using the AdS/CFT correspondence. The dual field theory is a thermal system in which initially a vacuum density for a non-conserved operator is turned on. We can see that in agreement with general thermal field theory expectation the system equilibrates: the expectation value of the non-conserved operator goes to zero exponentially and the entropy increases. In the field theory the process can be described quantitatively in terms of a thermofield state and exact agreement with the gravity answers is found.

Keywords: AdS/CFT Correspondence, Black Holes.

## Contents



## 1. Introduction

Recently, two of the present authors together with S. Hirano proposed a family of time dependent black hole solutions in 3 and 5 spatial dimensions that can be embedded into type IIB string theory [1]. In this paper we further interpret these solutions using the AdS/CFT correspondence [2–4], which relates the properties of the gravitational system to those of a field theory in one lower dimension. In the field theory we will demonstrate that the process we study corresponds to thermal equilibration. The black hole solution corresponds to a thermal bath in the field theory. On top of this thermal background initially the Lagrange density, which for a Maxwell field would be proportional to $\vec{E}^2 - \vec{B}^2$, has a non-trivial expectation value. While the energy density $\vec{E}^2 + \vec{B}^2$ is a conserved quantity, the Lagrange density is not and one would expect the system to thermalize, eventually partitioning the energy equally between electric and magnetic fields. On general grounds this return to equilibrium should be exponential with a characteristic thermalization time $\tau_{therm.}$. Since the process is dissipative,

the entropy should increase during thermalization. All these expectations are born out by explicit calculations.

Studying thermalization in any strongly coupled system is inherently difficult. The quark gluon liquid produced at RHIC, where the observed very short thermalization time clashes with weak coupling expectation, is an example where this issue is of practical interest. The AdS/CFT correspondence is a nice tool to analytically study certain solvable strongly coupled field theories and will hopefully serve to build our intuition about thermalization in strongly coupled systems. Earlier studies of thermalization using the AdS/CFT correspondence were either limited to small fluctuations around a thermal configuration, see e.g. [5–7], or approximate late time solutions [8]. In contrast, our gravity solution is an exact answer for all times with a large initial perturbation. The particular field theory we study, $N = 4$ super-Yang-Mills on a compact hyperbolic space at a fixed temperature, is too remote from the QCD fireball at RHIC to be of direct experimental relevance, but we find it remarkable that in this simple system the full thermalization process can be mapped out using the AdS/CFT correspondence. The thermalization time we find, $\tau_{therm.} = \frac{1}{2\pi T}$, has appeared before in other studies [5, 9, 10] where it played the role of a limiting value. For $T = 300$ MeV, this would give $\tau_{therm.} \sim 0.1$ fm/c. Reassuringly this is much faster than one would expect from a perturbative analysis.

Beyond the interest in studying thermalization our result is of notice in that it does not exhibit any sign of a Poincaré recurrence. This result demonstrates a clash between unitarity and the strict large number of color limit in the field theory. Such a clash is not unexpected, see [11, 12] for recent discussions.

In the following section we will review the time dependent black hole solution of [1]. In section 3 we calculate the properties of the dual field theory using the standard AdS/CFT dictionary. In section 4 we use the formalism of thermofield theory to construct a particular entangled state that encodes the exact density matrix of the dual field theory. In section 5 we use this thermofield state to calculate the expectation values of the Lagrangian and the Hamiltonian to leading order in the deformation parameter from the field theory side and establish agreement with the gravity calculation. In section 6 we discuss the implications our result has for the question of unitarity of the dual field theory.

## 2. Time dependent black holes

We begin with a scalar Einstein gravity described by

$$I = \frac{1}{16\pi G} \int d^d x \sqrt{g} \left( R - g^{ab} \partial_a \phi \partial_b \phi + (d-1)(d-2) \right) , \tag{2.1}$$

where the spacetime dimension $d$ is greater than or equals to three. As shown in Refs. [1,13], any solution of the above can be embedded into the type IIB supergravity for $d = 3$ and $d = 5$. The $d = 3$ ($d = 5$) solution describes the deformation of $\mathrm{AdS}_3 \times S^3$ ($\mathrm{AdS}_5 \times S^5$) geometry. Note that in three dimensions, $\phi$ is the IIB dilaton whereas $\sqrt{2}\phi$ corresponds to the dilaton for $d = 5$. We shall denote the dilaton field by $\tilde{\phi}$ including this extra normalization factor. The AdS radius is denoted by $l$, which we set to be unity.

The black hole solution describing a non-equilibrium thermal system was obtained in Ref. [1] using the method of so called Janus construction [13]. Its metric ansatz is taken as

$$ds^2 = f(\mu)(d\mu^2 + ds^2_{d-1}) \tag{2.2}$$

where $(d-1)$ dimensional metric $\bar{g}_{pq}$ satisfies

$$\bar{R}_{pq} = -(d-2)\bar{g}_{pq} \,. \tag{2.3}$$

The Einstein equations are reduced to

$$f' f' = 4f^3 - 4f^2 + \frac{4\gamma^2}{(d-1)(d-2)} f^{4-d} , \tag{2.4}$$

and the scalar equation is integrated once giving

$$\phi'(\mu) = \frac{\gamma}{f^{\frac{d-2}{2}}(\mu)} , \tag{2.5}$$

where $\gamma$ is the integration constant responsible for the Janus deformation. For $\gamma^2 \leq \gamma_c^2$ with

$$\gamma_c^2 = (d-2) \left( \frac{d-2}{d-1} \right)^{d-2} , \tag{2.6}$$

this can be solved by the integral [1]

$$\mu_0 \pm \mu = \int_f^\infty \frac{dx}{2\sqrt{x^3 - x^2 + \frac{\gamma^2}{(d-1)(d-2)} x^{4-d}}} , \tag{2.7}$$

where $\mu_0$ is chosen such that $\mu = 0$ at the turning point. One may show that $\mu_0 \geq \pi/2$. Defining $\phi_\pm$ by

$$\phi_\pm = \phi(\pm\mu_0) , \tag{2.8}$$

one finds that

$$\phi_+ - \phi_- = \int_{-\mu_0}^{\mu_0} d\mu \frac{\gamma}{f^{\frac{d-2}{2}}(\mu)}\,. \tag{2.9}$$

Now the trick is to take the metric $\bar{g}_{pq}$ as the cosmological form,

$$ds^2_{d-1} = -d\tau^2 + \cos^2\tau ds^2_\Sigma \tag{2.10}$$

where $ds^2_\Sigma$ is describing the compact, smooth, finite volume Einstein space metric in $(d-2)$ dimensions satisfying $R^\Sigma_{kl} = -(d-3)g^\Sigma_{kl}$. The coordinate $\tau$ is ranged over $[-\pi/2, \pi/2]$. For $d = 3$ case, the $\Sigma$ space corresponds to a circle $S^1$ and, for higher dimensions, the space can be given by the quotient of the hyperbolic space $H_{d-2}$ by a discrete subgroup of the hyperbolic symmetry group, $SO(1, d-2)$.

In summary the metric for the time dependent black hole is given by

$$ds^2 = f(\mu)(d\mu^2 - d\tau^2 + \cos^2\tau ds^2_\Sigma)\,. \tag{2.11}$$

The above form of the metric is suitable for the drawing of the Penrose diagram. The $\tau$ and $\mu$ coordinate can be used to represent the global structure of the spacetime. Since $\mu_0 \geq \pi/2$, the diagram is no longer a square but a rectangle elongated in the horizontal direction. The $\pm 45°$ lines describe the future and past horizons.

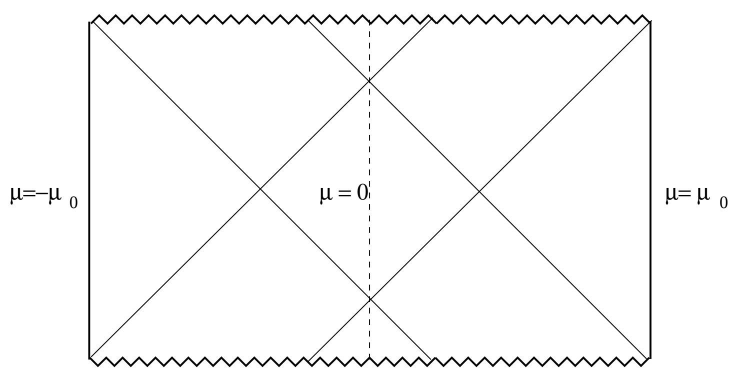


**Figure 1:** Penrose diagram for the time dependent black hole. The $\tau$ ($\in [-\pi/2, \pi/2]$) coordinate runs vertically upward and $\mu$ ($\in [-\mu_0, \mu_0]$) to the right horizontally.

For instance, the future horizon extended from the future infinity on the right hand side corresponds to a line $\mu - \mu_0 = \tau - \pi/2$. Representing the volume of $\Sigma$ space by $\mathcal{V}_\Sigma$, the future-horizon area is given by

$$A(\tau) = \mathcal{V}_\Sigma \Big[\cos(\tau) f^{\frac{1}{2}}(\mu_0 + \tau - \pi/2)\Big]^{d-2}\,. \tag{2.12}$$

One can check that the area is monotonically increasing as a function of $\tau$ starting from zero at $\tau = -\pi/2$ reaching the maximal value $\mathcal{V}_\Sigma$ at $\tau = \pi/2$. This is consistent with the area theorem of the black hole horizon.

To understand the geometry a little better it is instructive to look at the undeformed case. If the deformation parameter $\gamma$ is zero, the scalar field becomes a trivial constant and the metric

$$ds_0^2 = \frac{1}{\cos^2\mu}(d\mu^2 - d\tau^2 + \cos^2\tau ds_\Sigma^2) \tag{2.13}$$

describes the static black hole. The Penrose diagram for this case becomes a square and the physics of corresponding black hole is studied in Refs. [14–22]. Locally this space is just Anti-de Sitter space. The only difference to global AdS is that we had to perform an orbifold of the hyperbolic space $H_{d-2}$ to obtain the compact manifold $\Sigma$.

By the coordinate transformation,

$$\tanh t = \frac{\sin\tau}{\sin\mu}, \quad r = \frac{\cos\tau}{\cos\mu}, \tag{2.14}$$

the metric of the undeformed black hole may be brought to the form of the BTZ type [23],

$$ds_{eq}^2 = -(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + r^2 ds_\Sigma^2 \,. \tag{2.15}$$

The temperature of the black hole is $T_{eq} = \frac{1}{2\pi}$ which is in the unit of the AdS radius. The horizon is at $r = 1$ and the corresponding black hole entropy is

$$S_{eq} = \frac{\mathcal{V}_\Sigma}{4G} \,. \tag{2.16}$$

The mass of the black hole is evaluated as [20]

$$M_{eq} = \frac{\mathcal{V}_\Sigma}{8\pi G}\left(\frac{d-2}{d-1}\right)\left(\frac{d-3}{d-1}\right)^{\frac{d-3}{2}} \,. \tag{2.17}$$

As explained in [21] the value $T = \frac{1}{2\pi}$ is also special in the dual field theory on a hyperbolic space, since this particular thermal state can be formally obtained from the vacuum of the Einstein universe. At other values of the temperature the dual black hole is no longer locally AdS, but instead is given by a $k = -1$ Schwarzschild black hole with a non-trivial mass parameter, which for the $k = -1$ slicing formally is negative for temperatures less than the special $T = \frac{1}{2\pi}$. To find time dependent solutions for any temperature other than $T = \frac{1}{2\pi}$ one has to turn on an explicit time dependence in the dilaton, $\phi(\mu, \tau)$, and the metric function $f(\mu, \tau)$. While such a solution can at least be obtained for small deformation parameter $\gamma$ or small temperature difference $T = \frac{1}{2\pi} + \delta T$ by perturbing around the known solutions, we will limit our analysis in this paper to the fully solvable case of $T = \frac{1}{2\pi}$.

Since the effect of our time dependent deformation vanishes at late times, these thermodynamic properties of the undeformed case will describe the deformed solution at late

times when the system returns to equilibrium. The time dependent solution describes non-equilibrium physics and we have to carefully determine the energy density and the entropy independently and cannot rely on the first law of thermodynamics to relate them. At least the energy density can be reliably defined even in the time dependent context using the holographic stress-energy tensor, and we will determine it in the next section where we discuss the gauge/gravity correspondence.

## 3. Correspondence

In this section we like to discuss the physics of the super Yang-Mills (SYM) theory dual to the time dependent black hole. Since we are dealing with the deformation of the AdS/CFT correspondence, we shall use the standard framework given in [3, 4] for the interpretation of the geometry of the time dependent black hole. Namely the on-shell supergravity action with given boundary sources is providing the standard generating functional of connected correlators of operators dual to the sources. In this framework, the classical geometry and the bulk spacetime have a natural dual Yang-Mills theory interpretation in the planar large $N_c$ limit.

First by choosing the conformal factor $h^2 = \cos^2 \tau / f(\mu)$, the boundary metric for the CFT is given by [1]

$$ds_B^2 = -dt^2 + ds_\Sigma^2 \,. \tag{3.1}$$

There are two separated boundaries at $\mu = \pm\mu_0$ and the boundary time $t$ ($\in (-\infty, \infty)$)is related to $\tau$ by $\tanh t = \pm \sin \tau$ respectively at each boundary.

For $d = 5$ case, the $N = 4$ SYM theory on the above boundary spacetime is the corresponding dual system. Since the values of the dilaton on the two boundaries are different from each other, the corresponding CFT's of the two boundaries now become different as a result of the deformation. The number of colors $N_c$ agrees with each other while the 't Hooft couplings $\lambda_\pm$ become different by the time dependent deformation[1]. The situation in the $d = 3$ case is not much different. The CFT Lagrangian density is proportional to the inverse of the string coupling by

$$\mathcal{L}_\pm \;\propto\; (g_s^\pm)^{-1} = e^{-\tilde{\phi}_\pm} \,. \tag{3.2}$$

The scalar field behaves, in the near boundary region, as

$$\phi \sim \phi_\pm \mp \frac{\gamma}{d-1} |\mu \mp \mu_0|^{d-1} + \cdots \,. \tag{3.3}$$

[1]The 't Hooft coupling $\lambda$ is related to the string coupling and the number of color by $\lambda = 4\pi g_s N_c$.

We shall be discussing the behavior of the CFT from the view point of the right hand side boundary at which the upper sign in the above is relevant. The system on the left hand side can be treated in a similar manner. Noting that

$$h \sim |\mu - \mu_0| / |\cos \tau| = |\mu - \mu_0| \cosh t \tag{3.4}$$

near boundary region of the right hand side, the near boundary behavior of the scalar can be presented as

$$\phi \sim \phi_+ - \frac{\gamma}{(d-1)\cosh^{d-1} t} h^{d-1} + \cdots . \tag{3.5}$$

Since the operator dual to the dilaton is the CFT Lagrange density, one is led to

$$\langle \mathcal{L} \rangle = \frac{\tilde{\gamma}}{8\pi G} \frac{1}{\cosh^{d-1} t} , \tag{3.6}$$

where $\tilde{\gamma}$ equals to $\sqrt{2}\gamma$ for $d = 5$ and to $\gamma$ in $d = 3$ as long as we use the standard definitions of the field theory Lagrangians. For other dimensions no string theory embedding of the Janus geometries and no dual field theory have been proposed so far, so we will assume that in all those cases the Lagrange density of the dual field theory is scaled in such a way that $\tilde{\gamma} = \gamma$.

By studying the near boundary behavior of the metric tensor, one can obtain the expectation value of the boundary energy momentum tensor. We follow the holographic renormalization methods in Ref. [24] and the result is

$$\begin{aligned} \langle T_{00} \rangle &= \frac{1}{8\pi G} \left( \frac{d-2}{d-1} \right) \left( \frac{d-3}{d-1} \right)^{\frac{d-3}{2}} \\ \langle T_{ij} \rangle &= \frac{1}{8\pi G} \left( \frac{1}{d-1} \right) \left( \frac{d-3}{d-1} \right)^{\frac{d-3}{2}} h_{ij} , \end{aligned} \tag{3.7}$$

where $h_{ij}$ is the metric tensor for the $\Sigma$ space. The total energy of the system $E$ agrees with the equilibrium value $M_{eq}$. The result is consistent with the tracelessness condition of the energy momentum tensor, i.e. $T^{\mu}{}_{\mu} = 0$, which is due to the conformal symmetry.

There is one important clarification about the above computation. When we calculate the operator expectation values or more generally correlators using the on-shell gravity action, we are not using the Lorentzian geometries. If we were working in the Lorentzian signature, we would be troubled with the singularities in solving the gravity equations. Instead one computes correlators using Euclidean geometries. As will be shown in the next section, the Euclidean black hole geometry is perfectly smooth and regular everywhere. There is no notion of horizon there. Hence the gravity equation with specified boundary sources is well defined and gives a unique solution. Then using the corresponding on-shell action, one can compute

the Euclidean correlators of the boundary CFT operators. One then gets the Lorentzian-signature correlators by the Wick rotation (in the boundary CFT) or by an appropriate analytic continuation (on the geometric side).

From the above behavior of the expectation values, it is clear that we are dealing with a quantum system whose state is not stationary. The system is homogeneous over the finite-volume $\Sigma$ space, which explains the time independence of the boundary energy momentum tensor. As the black hole possesses the $Z_2$ time reversal symmetry $\tau \to -\tau$, the same is true for the boundary system, which is symmetric under the time reversal $t \to -t$, too.

The system starts at $t = 0$ from an out-of-equilibrium state where the kinetic energy differs from the potential energy. Then this out-of-equilibrium situation settles down as time goes by reaching exponentially the equilibrium state where the kinetic energy equals to the potential energy. The exponential approach of the equilibrium can be seen clearly from the late time behavior of the expectation value of the Lagrange density. Recalling that we work at a temperature of $\frac{1}{2\pi}$ in units where the curvature radius of the hyperbolic space is 1, the thermalization time can be written as $\tau_{therm.} = \frac{1}{2\pi T}$. The final entropy of the system is given by $S = S_{eq}$, which should be larger than the initial entropy $S_0$ at $t = 0$.

A few comments are in order. Since we are dealing with a thermal equilibration process, the first law does not have to hold. Namely $TdS \neq dE + pdV$. Since the total energy and volume of the system are constant in time, the right hand side is zero whereas the left hand side (if defined) cannot be vanishing because of the change of the entropy. This is not a problem since the system is not in a quasi-equilibrium state. Even the second law is not working since the system has the $Z_2$ symmetry and, the entropy should be decreasing as a function of time for $t < 0$ reaching the minimal value at $t = 0$. However for the finite entropy system this kind of fine tuning at $t = -\infty$ is not totally impossible.

From the geometry we have already computed the horizon area as a function of $\tau$ which is related to the boundary time by $\tanh t = \sin \tau$. Along the future horizon on the right hand side, the horizon area is monotonically increasing as we discussed before. But for the time dependent case, we do not have a formalism to relate the horizon area to the entropy or some other quantity of the boundary CFT. Similarly the temperature as a function of time cannot be computed from the geometry because there is no notion of the periodicity of the Euclidean thermal circle for the time dependent case. Since the system is not in a thermal equilibrium, we do not know how to define the temperature of the system either.

Finally let us describe one failure of the geometric description. The late time behavior of the expectation value of the Lagrange density computed from geometry appears extremely

natural from the physics view point of the thermal equilibration. However this behavior is not consistent with the quantum Poincaré recurrence theorem [25–28]. The theorem states that for any quantum system the wave function or expectation values of operators of the system will continuously return arbitrarily closely to their initial values in a finite amount of time once the spectrum is discrete. Since the boundary quantum system indeed has the discrete spectrum, one can see that there is a failure in the geometric description once we accept the AdS/CFT correspondence.

As discussed Ref. [11], this failure of the geometrical description is of order $e^{-aS_{eq}}$ with some order-one constant $a$. The discrepancy then becomes relevant around $t \sim aS_{eq}l$. Since $S_{eq} \propto N_c^2 = \lambda^2/(4\pi g_s)^2$ for $d = 5$, one can see that the effect is nonperturbative in its nature.

Finally as discussed in Refs. [22,29], the $d = 5$ geometry shows a nonperturbative instability corresponding to $D3 - \overline{D3}$ pair creation. The same instabilities are present in the $N = 4$ SYM theory on the $\Sigma$ space which is negatively curved. Hence the correspondence is still working in this respect. The instabilities can be suppressed by taking the volume $\mathcal{V}_\Sigma$ large or the string coupling $g_s$ small. Since we are in the decoupling large $N_c$ limit with $\lambda$ fixed, the instabilities can be ignored and do not affect any discussions above.

## 4. Construction of the thermofield state

The time dependent black hole solution allows an analytic continuation, $\tau = -i\tau_E$, leading to the Euclidean geometry,

$$ds_E^2 = f(\mu)(d\mu^2 + d\tau_E^2 + \cosh^2 \tau_E ds_\Sigma^2) \tag{4.1}$$

with the scalar field $\phi(\mu)$ intact. $\tau_E$ is ranged over $(-\infty, \infty)$. The Euclidean geometry is smooth everywhere and has a boundary. The conformal shape of the $(\mu, \tau_E)$ space is a disk as the case of the usual time-independent black hole. In this section we shall provide the physical interpretation of the above Euclidean geometry in terms of thermofield dynamics [30–33].

For the $4d$ Poincarè-invariant field theories, their instanton solution possesses $O(4)$ invariance and let us take $t_E = 0$ as the fixed point of the $Z_2$ symmetry $t_E \to -t_E$ where $t_E$ is the Euclidean time. At this point the time derivative of fields vanish again due to the $Z_2$ symmetry. This $t_E = 0$ field configuration may be interpreted as an initial configuration from which the Lorentzian dynamics follows. The subsequent Lorentzian dynamics for $t \geq 0$ can be obtained from the instanton solution by analytic continuation [34]. At $t_E = t = 0$, the Lorentzian and the Euclidean configurations agree with each other and the time derivatives (velocities) of both fields vanish, which helps them join smoothly. Thus at

least semi-classically we conclude that the Euclidean solution provides an initial state for the Lorentzian time evolution.

In case of geometry, this procedure corresponds to the Hartle-Hawking construction of the wave function [35]. In our problem we shall follow the proposal of Ref. [11] to construct the corresponding thermofield initial state. Namely we patch the half of Euclidean geometry sliced at $\tau_E = 0$ to the upper half of the Lorentzian solution sliced at $\tau = 0$. Since the geometry involves two boundaries, the corresponding Hilbert space consists of $\mathcal{H} = \mathcal{H}_+ \times \mathcal{H}_-$. Unlike the conventional thermofield formalism, the two Hamiltonians $H_+$ and $H_-$ differ from each other, which is responsible for the time dependence of a single boundary description as we will show below. According to the proposal in Ref. [11], the Euclidean geometry defines a boundary Hamiltonian and allows a Euclidean boundary time evolution, which will determine the initial thermofield state by

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_{mn} \langle E^+_m | U | E^-_n \rangle \; |E^+_m\rangle \times |E^-_n\rangle \tag{4.2}$$

where $Z$ is the normalization factor. The Euclidean evolution operator $U$ is given by

$$U = T \exp\Big[ - \int_{s_-}^{s_+} ds\, H(s) \Big] \tag{4.3}$$

where $s$ is the boundary Euclidean time with $s_\pm$ denoting the two boundary times at $\tau_E = 0$. Since the two boundary Hamiltonians differ from each other, $H(s)$ becomes time dependent. In this respect, the above is a small generalization of the Maldacena's proposal but this naturally follows from the fact that the Euclidean boundary Hamiltonian is now time dependent.

The boundary of the metric (4.1) can be identified by the fact that the scale factor $f(\mu)\cosh^2 \tau_E$ is infinitely large on the boundary. The boundary is then $\mu = \pm\mu_0$ and $\tau_E = \pm\infty$. In $(\mu, \tau_E)$ space, $(\mu, \pm\infty)$ become two points on the boundary, which are antipodal. At these points, the segments $\mu = \pm\mu_0$ are joined to form a complete circle. The conformal shape of $(\mu, \tau_E)$ space is depicted in Figure 2.

In the previous section, we have introduced the boundary time $t$ by $\tanh t = \pm \sin \tau$ respectively for $\mu = \pm\mu_0$. By the analytic continuation $t = -it_E$, the Euclidean boundary time $t_E$ is related to $\tan t_E = \pm \sinh \tau_E$. For $\mu = \mu_0$ ($\mu = -\mu_0$), $t_E$ is chosen to run over $[\pi/2, 3\pi/2]$ ($[-\pi/2, \pi/2]$) from $B$ ($A$) to $A$ ($B$). In the lower half part of the Euclidean geometry, the boundary time ranges over $[0, \pi]$. The boundary Hamiltonian is identified as $H_\pm$ (that is obtained from $\mathcal{L}_\pm$ of the previous section) respectively for $\mu = \pm\mu_0$. The evolution operator $U$ then becomes

$$U = e^{-\frac{\pi}{2} H_+} e^{-\frac{\pi}{2} H_-} = e^{-\frac{\beta_{eq}}{4} H_+} e^{-\frac{\beta_{eq}}{4} H_-} \,, \tag{4.4}$$

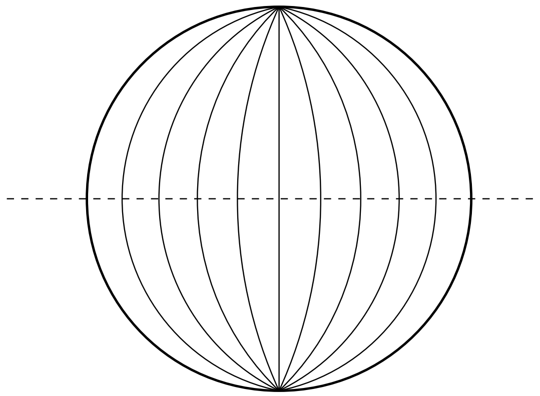


**Figure 2:** The conformal diagram of the Euclidean solution in $(\mu,\,\tau_E)$ space. The curves represent constant $\mu$ lines. Along the curves, $\tau_E$ runs from $-\infty$ at $B$ to $+\infty$ at $A$ in the upward direction. The right (left) half of boundary corresponds to $\mu = \mu_0$ ($\mu = -\mu_0$). The dotted line is the $\tau_E = 0$ line and the lower half is used to construct the thermofield state.

where we have used $\beta_{eq} \equiv 1/T_{eq} = 2\pi$.

Therefore the thermofield initial state becomes

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_{mn} \langle E_m^+ | E_n^- \rangle e^{-\frac{\beta_{eq}}{4}(E_m^+ + E_n^-)} \, |E_m^+\rangle \times |E_n^-\rangle \,. \tag{4.5}$$

The state becomes the usual one if $H_+ = H_-$.

Since two boundary CFT's are independent, the generic time evolution involves two boundary times $t_+$ and $t_-$ with the Hamiltonians $H_+$ and $H_-$ respectively. More explicitly,

$$|\Psi(t_+, t_-)\rangle = e^{-i(t_+ H_+ \times I + t_- I \times H_-)} |\Psi(0,0)\rangle \,. \tag{4.6}$$

If $H_+ = H_-$, the thermofield initial state is invariant under $\tilde{H} = H_+ \times I - I \times H_-$. But for the present case with deformation, we do not have this symmetry any more and this will be the reason for the time dependence of the thermal system.

The density matrix $\rho_+$ for the boundary system on the right hand side is given by

$$\rho_+ = \mathrm{tr}_- |\Psi\rangle\langle\Psi| \tag{4.7}$$

where $\mathrm{tr}_\pm$ denotes the trace over the $\mathcal{H}_\pm$ Hilbert space. If $O_+$ is any operator defined in the $\mathcal{H}_+$ space, the thermal expectation values are defined by

$$\langle O_+ \rangle = \mathrm{tr}_+ \rho_+ O_+ \,. \tag{4.8}$$

This description of the system by the density matrix provides us with the single boundary view.

The density matrix $\rho_+$ is no longer commuting with $H_+$ and, hence, time dependent. Then the expectation value $\langle O_+ \rangle$ is in general time dependent, which is consistent with our result of the previous section for $O_+ = \mathcal{L}$.

Note that the pure state expectation value, $\langle \Psi | O_+ | \Psi \rangle$, is also giving the expectation operator $\langle O_+ \rangle$. Therefore we conclude that the above density matrix (or the thermofield state) of the boundary CFT is corresponding to the geometry of the time dependent black hole if one ignores the failure of the previous section that is nonperturbative.

## 5. Check for the thermofield state

In order to verify our proposal for the thermofield state we would like to compute the expectation value of the Lagrangian and the Hamiltonian in the field theory and compare with the supergravity answers in equations (3.6) and (3.7) to leading order in the deformation parameter $\gamma$. As in [1,36] one can employ the techniques of conformal perturbation theory to calculate field theory expectation values (suitably generalized to the non-trivial thermofield state) in a power series in $\gamma$. The expectation value of the Lagrangian in (3.6) is exactly linear to all orders in $\gamma$. The expectation value of the Hamiltonian that one gets by integrating the energy density $T_{00}$ in (3.7) is the exact answer as well. Independent of $\gamma$ one finds the equilibrium values, so that the leading correction to $\langle H \rangle$ is zero in the gravity calculation.

To compute the expectation values in the field theory we start from the thermofield state (4.5). Note that in the following we specialize to the case $d = 3$. For the expectation value of the Lagrange density $\langle \mathcal{L}_+(t, 0) \rangle$ we get

$$\langle \mathcal{L}_+(t,0) \rangle = \frac{1}{Z} \mathrm{tr} \mathcal{L}_+(t,0) e^{-\frac{\beta_{eq}}{4} H_+} e^{-\frac{\beta_{eq}}{2} H_-} e^{-\frac{\beta_{eq}}{4} H_+} \tag{5.1}$$

where $\beta_{eq} = 2\pi$ in our case. Let $H_- - H_+ = \delta H$ and we expand $e^{-\pi(H_+ + \delta H)}$ by

$$e^{-\pi(H_+ + \delta H)} = e^{-\pi H_+} - \int_0^\pi d\tau e^{-(\pi - \tau) H_+} \delta H e^{-\tau H_+} + \cdots \tag{5.2}$$

Then the leading term

$$\mathrm{tr}\, \mathcal{L}_+(t,0) e^{-2\pi H_+} \tag{5.3}$$

is vanishing. The remaining contribution gives

$$\langle \mathcal{L}_+(t,0) \rangle = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \frac{1}{Z} \mathrm{tr}\, \mathcal{L}_+(t,0)\ \delta H(-i(\tau - \pi)) e^{-2\pi H_+} + \cdots \tag{5.4}$$

This can be arranged as[2]

$$\langle \mathcal{L}_+(t,0)\rangle = (e^{\phi_+}/e^{\phi_-} - 1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \int_0^{2\pi} d\theta \langle \mathcal{L}_+(t,0)\mathcal{L}_+(-i(\tau-\pi),\theta)\rangle_{\gamma=0}\,. \tag{5.5}$$

Now let us use the formula

$$\langle \mathcal{L}_+(t_1,\theta_1)\mathcal{L}_+(t_2,\theta_2)\rangle_{\gamma=0} = \frac{1}{16\pi G}\frac{4}{\pi}\frac{1}{4}\sum_{m=\infty}^{\infty}\frac{1}{[\cosh(t_1-t_2)-\cosh(\theta_1-\theta_2+2\pi m)-i\epsilon]^2} \tag{5.6}$$

from equation (2.5) of Ref. [11]. We have fixed the normalization by comparing to the standard AdS/CFT result for the 2-point function for the operator $O$ dual to a scalar with kinetic term $-\frac{\eta}{2}(\partial\phi)^2$

$$\langle O(x)O(0)\rangle = \eta\frac{2\Delta-d}{\pi^{d/2}}\frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)}\frac{1}{x^{2\Delta}} = \frac{1}{16\pi G}\frac{4}{\pi}\frac{1}{x^4} \tag{5.7}$$

considering the limit $t_1 \to t_2$ and $\theta_1 \to \theta_2$. The factor $1/4$ is introduced to cancel the square of the coefficient $1/2$ of $\cosh x - 1 = \frac{1}{2}x^2 + \cdots$.

Therefore one finally has

$$\langle \mathcal{L}_+(t,0)\rangle = \frac{1}{16\pi G}\frac{2\gamma}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \int_{-\infty}^{\infty} d\theta \frac{1}{[\cosh(t+i\tau)+\cosh\theta]^2} \tag{5.8}$$

where we have used $(e^{\phi_+}/e^{\phi_-} - 1) = 2\gamma + \cdots$. To evaluate the integral, we first note that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\tau}{a\cos\tau + ib\sin\tau + c} = 2\frac{\tan^{-1}\left(\frac{c-a+ib}{\sqrt{c^2-a^2+b^2}}\right)+\tan^{-1}\left(\frac{c-a-ib}{\sqrt{c^2-a^2+b^2}}\right)}{\sqrt{c^2-a^2+b^2}} = 2\frac{\tan^{-1}\left(\frac{\sqrt{c^2-a^2+b^2}}{a}\right)}{\sqrt{c^2-a^2+b^2}} \tag{5.9}$$

where, for the last equality, we use $\tan(A+B) = (\tan A + \tan B)/(1-\tan A\tan B)$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \frac{1}{[\cosh(t+i\tau)+\cosh\theta]^2} &= 4\int_0^{\infty} d\theta \left(\frac{-d}{\sinh\theta d\theta}\right)\frac{\tan^{-1}\left(\frac{\sinh\theta}{\cosh t}\right)}{\sinh\theta} \\ &= \frac{-4}{\cosh^2 t}\int_0^{\infty}\frac{dw}{w}(\tan^{-1}w/w)'\,. \end{aligned}\tag{5.10}$$

[2]In a Hamiltonian framework the propagator can be given a path integral definition $\langle x'|e^{-\tau H'}|x\rangle = \int_{x(0)=x}^{x(\tau)=x'}\mathcal{D}xe^{-S'_E}$ where $S'_E$ is the Euclidean action $S'_E = (1+\delta)S_E = -(1+\delta)\int_0^\tau ds\mathcal{L}(-is)$ and $S_E$ is the Euclidean action in the undeformed theory. Then

$$\langle x'|e^{-\tau H'}|x\rangle = \int_{x(0)=x}^{x(\tau)=x'}\mathcal{D}xe^{-S_E}(1+\delta\int_0^\tau \mathcal{L}(-is)) + \cdots = \langle x'|e^{-\tau H}|x\rangle + \delta\langle x'|e^{-\tau H}\int_0^\tau ds\hat{\mathcal{L}}(-is)|x\rangle + \cdots$$

where $\hat{\mathcal{L}}(-is) = e^{sH}(K-V)e^{-sH}$ with $H = K+V$. $K$ is the kinetic energy operator while $V$ is for the potential operator. So the perturbation is really in terms of the Lagrange operator $(K-V)$ with imaginary time evolution.

The definite integral is evaluated as

$$-2\int_0^\infty \frac{dw}{w}\left(\frac{\tan^{-1}w}{w}\right)' = \frac{w+(w^2-1)\tan^{-1}w}{w^2}\bigg|_0^\infty = \pi/2\,. \tag{5.11}$$

Hence we get

$$\langle \mathcal{L}_+(t,0)\rangle = \frac{1}{8\pi G}\frac{\gamma}{\cosh^2 t}\,, \tag{5.12}$$

in complete agreement with the gravity result in (3.6).

In an analogous fashion we can calculate the order $\gamma$ correction to the expectation value of the the boundary energy momentum tensor $\langle T_{\mu\nu}\rangle$. The calculation proceeds as above, however this time the relevant 2-point function is $\langle T_{\mu\nu}\mathcal{L}\rangle$ which vanishes (and hence the correction vanishes, too). This is in agreement with the supergravity result (3.7) which states that to all orders in $\gamma$ the expectation value of $T_{\mu\nu}$ is given by the equilibrium answer, so in particular the order $\gamma$ correction vanishes.

## 6. Discussion

In this paper we have investigated a time dependent black hole solution utilizing the AdS/CFT correspondence. We have found an exact solution of the supergravity in the large $N_c$ and large AdS curvature radius limit, which corresponds to a large initial perturbation of the black hole geometry and have observed an exponential return to equilibrium.

We constructed the thermofield state from the Hartle-Hawking wavefunction and showed the agreement on both sides of the duality for the time dependent expectation value of the Lagrangian density to lowest order in conformal perturbation theory. Hence the time dependent black holes spacetimes, we have discussed, yield an interesting laboratory to study equilibration of strongly coupled gauge theories. A more detailed study of these spacetimes, including the five dimensional case, the calculation of higher point correlation functions and higher orders in conformal perturbation theory would be interesting. Unfortunately, the nature of the Janus ansatz implies that the gauge theories are defined on compact spaces of negative curvature, which is not the case one is most interested in for "real world" applications.

In Ref. [11] it was found that a small perturbation around the eternal black hole leads to exponentially decaying correlation functions, which are inconsistent with the quantum Poincaré recurrence theorem and hence with unitarity. The operator expectation values we computed from the on-shell supergravity action are once more inconsistent with the quantum Poincaré recurrence theorem. As in Ref. [11] this failure of the correspondence in the large time limit can be shown to occur as a nonperturbative effect. Therefore this failure is not a

contradiction at all. In the strict planar limit where the evaluation in the dual supergravity theory in terms of the classical on-shell action is valid the unitarity of the field theory is not manifest. It only gets reinstated by considering exponentially suppressed corrections.

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## References

[1] D. Bak, M. Gutperle and S. Hirano, “Three dimensional Janus and time-dependent black holes,” JHEP **0702**, 068 (2007) [arXiv:hep-th/0701108].

[2] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113] [arXiv:hep-th/9711200].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[5] P. K. Kovtun and A. O. Starinets, “Quasinormal modes and holography,” Phys. Rev. D **72**, 086009 (2005) [arXiv:hep-th/0506184].

[6] R. A. Janik and R. Peschanski, “Gauge / gravity duality and thermalization of a boost-invariant perfect fluid,” Phys. Rev. D **74**, 046007 (2006) [arXiv:hep-th/0606149].

[7] J. J. Friess, S. S. Gubser, G. Michalogiorgakis and S. S. Pufu, “Expanding plasmas and quasinormal modes of anti-de Sitter black holes,” JHEP **0704**, 080 (2007) [arXiv:hep-th/0611005].

[8] D. Bak and R. A. Janik, “From static to evolving geometries: R-charged hydrodynamics from supergravity,” Phys. Lett. B **645**, 303 (2007) [arXiv:hep-th/0611304].

[9] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L. G. Yaffe, “Energy loss of a heavy quark moving through N = 4 supersymmetric Yang-Mills plasma,” JHEP **0607**, 013 (2006) [arXiv:hep-th/0605158].

[10] C. P. Herzog, “Energy loss of heavy quarks from asymptotically AdS geometries,” JHEP **0609**, 032 (2006) [arXiv:hep-th/0605191].

[11] J. M. Maldacena, “Eternal black holes in Anti-de-Sitter,” JHEP **0304**, 021 (2003) [arXiv:hep-th/0106112].

[12] G. Festuccia and H. Liu, "The arrow of time, black holes, and quantum mixing of large N Yang-Mills theories," arXiv:hep-th/0611098.

[13] D. Bak, M. Gutperle and S. Hirano, "A dilatonic deformation of AdS(5) and its field theory dual," JHEP **0305**, 072 (2003) [arXiv:hep-th/0304129].

[14] R. B. Mann, "Pair production of topological anti-de Sitter black holes," Class. Quant. Grav. **14** (1997) L109 [arXiv:gr-qc/9607071].

[15] M. Banados, "Constant curvature black holes," Phys. Rev. D **57**, 1068 (1998) [arXiv:gr-qc/9703040].

[16] D. R. Brill, J. Louko and P. Peldan, "Thermodynamics of (3+1)-dimensional black holes with toroidal or higher genus horizons," Phys. Rev. D **56** (1997) 3600 [arXiv:gr-qc/9705012].

[17] L. Vanzo, "Black holes with unusual topology," Phys. Rev. D **56** (1997) 6475 [arXiv:gr-qc/9705004].

[18] R. B. Mann, "Topological black holes: Outside looking in," arXiv:gr-qc/9709039.

[19] J. D. E. Creighton and R. B. Mann, "Entropy of constant curvature black holes in general relativity," Phys. Rev. D **58**, 024013 (1998) [arXiv:gr-qc/9710042].

[20] D. Birmingham, "Topological black holes in anti-de Sitter space," Class. Quant. Grav. **16**, 1197 (1999) [arXiv:hep-th/9808032].

[21] R. Emparan, "AdS/CFT duals of topological black holes and the entropy of zero-energy states," JHEP **9906**, 036 (1999) [arXiv:hep-th/9906040].

[22] A. Buchel, "Gauge theories on hyperbolic spaces and dual wormhole instabilities," Phys. Rev. D **70**, 066004 (2004) [arXiv:hep-th/0402174].

[23] M. Banados, C. Teitelboim and J. Zanelli, "The Black hole in three-dimensional space-time," Phys. Rev. Lett. **69**, 1849 (1992) [arXiv:hep-th/9204099].

[24] K. Skenderis, "Lecture notes on holographic renormalization," Class. Quant. Grav. **19**, 5849 (2002) [arXiv:hep-th/0209067].

[25] P. Boccieri and A. Loinger, "Quantum recurrence theorem", Phys. Rev. **107**, 337 (1957).

[26] L.S. Schulman, "Note on the quantum recurrence theorem", Phys. Rev. A **18**, 2379 (1978)

[27] I.C. Percival, "Almost Periodicity and the Quantal H Theorem", J. Math. Phys. **2**, 235 (1961).

[28] L. Dyson, M. Kleban and L. Susskind, "Disturbing implications of a cosmological constant," JHEP **0210**, 011 (2002) [arXiv:hep-th/0208013].

[29] E. Witten and S. T. Yau, Adv. Theor. Math. Phys. **3**, 1635 (1999) [arXiv:hep-th/9910245].

[30] J. S. Schwinger, "Brownian motion of a quantum oscillator," J. Math. Phys. **2** (1961) 407.

[31] P. M. Bakshi and K. T. Mahanthappa, Expectation Value Formalism in Quantum Field Theory, 1, J. Math. Phys. **4** (1963) 4.

[32] L. V. Keldysh, "Diagram technique for nonequilibrium processes," Zh. Eksp. Teor. Fiz. **47** (1964) 1515 [Sov. Phys. JETP **20** (1965) 1018].

[33] Y. Takahasi and H. Umezawa, "Thermo Field Dynamics," Collect. Phenom. **2** (1975) 55.

[34] C. G. . Callan and S. R. Coleman, "The Fate Of The False Vacuum. 2. First Quantum Corrections," Phys. Rev. D **16**, 1762 (1977).

[35] J. B. Hartle and S. W. Hawking, "Wave Function Of The Universe," Phys. Rev. D **28**, 2960 (1983).

[36] A. B. Clark, D. Z. Freedman, A. Karch and M. Schnabl, "The dual of Janus ((<:) < −− > (:>)) an interface CFT," Phys. Rev. D **71**, 066003 (2005) [arXiv:hep-th/0407073].